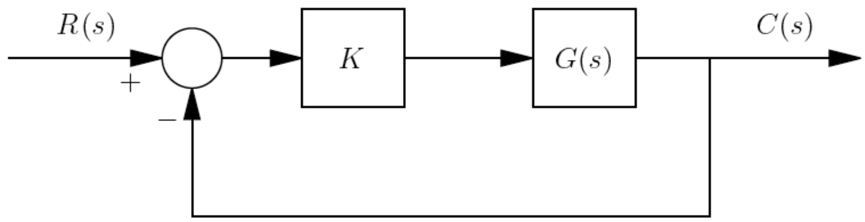
Root Locus

- Root locus definition
- Root locus sketching rules
- Examples

- Root locus is no longer used much as a computational tool
- But, the perspective of how poles move as influenced by loop singularities is still extremely useful as a guide for design

Configuration for studying root locus



One pole

If $G(s) = \frac{1}{s}$, the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{\dfrac{K}{s}}{1 + \dfrac{K}{s}},$$

with one pole at $s = -K$. We present this information in the form of a *root-locus diagram* as shown in Fig. 3.2.

The step response for this system is

$$c(t) = 1 - \mathrm{e}^{-\frac{t}{K}} \qquad , \qquad t > 0$$

and is stable for all $K > 0$. As we increase K, the system becomes proportionally faster with no loss of stability. This seems too good to be true, and as a practical matter, it is!

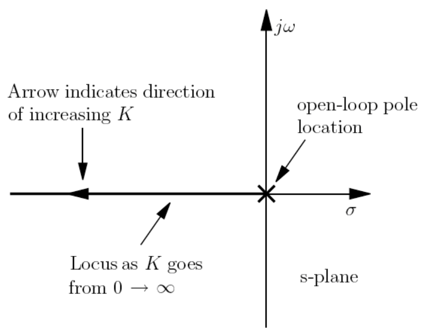


Figure 3.2: Root locus diagram for system with one pole.

Two poles

Next, let

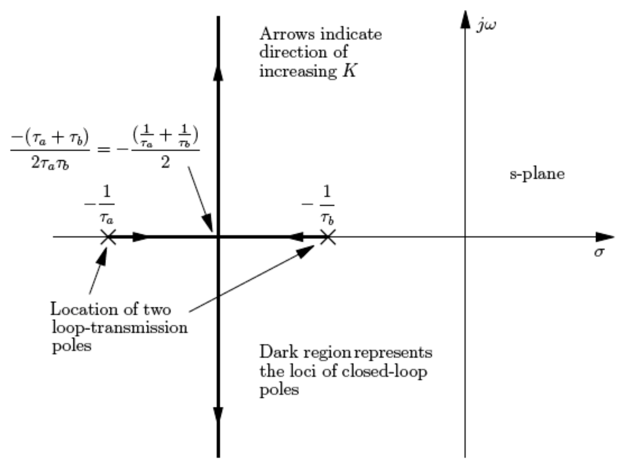
$$G(s) = \frac{1}{(\tau_a s + 1)(\tau_b s + 1)}$$

For this loop-transmission, the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{\dfrac{K}{(\tau_a s + 1)(\tau_b s + 1)}}{1 + \dfrac{K}{(\tau_a s + 1)(\tau_b s + 1)}} = \frac{K}{\tau_a \tau_b s^2 + (\tau_a + \tau_b)s + 1 + K}$$

The closed-loop poles are located at

$$s_1, s_2 = \frac{-(\tau_a + \tau_b) \pm \sqrt{(\tau_a + \tau_b)^2 - 4(1+K)\tau_a \tau_b}}{2\tau_a \tau_b}$$



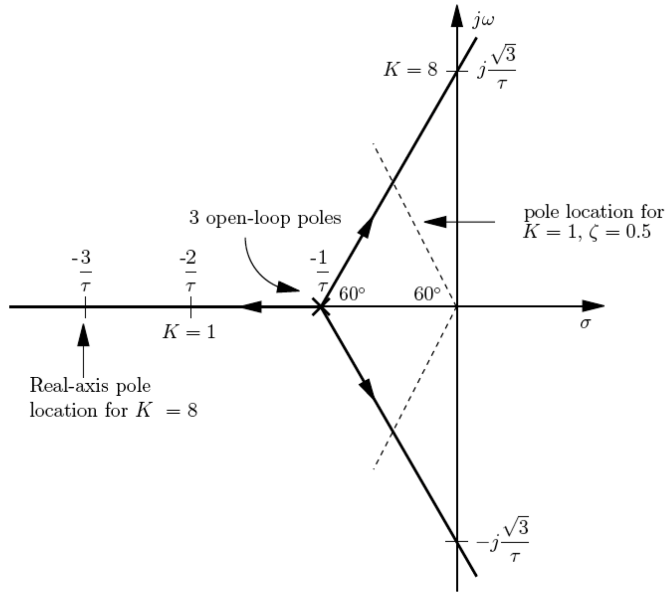
Three poles

We have also looked at a system with three coincident open-loop poles,

$$G(s) = \frac{1}{(\tau s + 1)^3}$$

The characteristic equation for this system is $\tau^3 s^3 + 3\tau^2 s^2 + 1 + K = 0$. Factoring for various K yields:

- $K = 0$, 3 poles @ $s = -\frac{1}{\tau}$
- $K = 1$, poles @ $s = -\frac{2}{\tau}$, and $-\frac{1}{2\tau} \pm j\frac{\sqrt{3}}{2\tau}$
- $K = 8$, poles @ $s = -\frac{3}{\tau}$, and $\pm j\frac{\sqrt{3}}{\tau}$
- $K = 64$, poles @ $s = -\frac{5}{\tau}$, and $\frac{1}{\tau} \pm j\frac{2\sqrt{3}}{\tau}$



Our standard system has the form shown in Fig. 3.4. We assume the number of poles

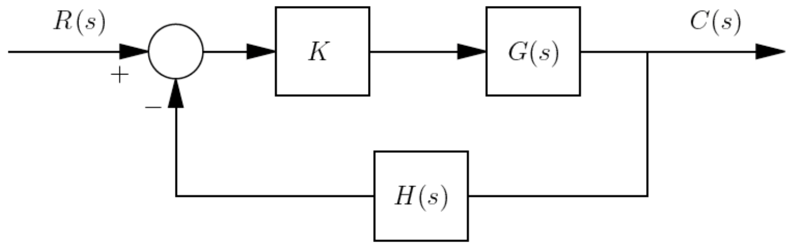


Figure 3.4: Block diagram for studying root locus.

of $G(s)H(s)$ is equal to or greater than the number of zeros since we are dealing with physically realizable systems that do not have unbounded high frequency gain.

When $G(s)H(s)$ is written in the form

$$G(s)H(s) = K \frac{\prod_{i=1}^{z} (s - z_i)}{\prod_{i=1}^{p} (s - p_i)}$$

the gain K is called the **root locus gain.** We assume that K accounts for the entire multiplicative gain associated with $KG(s)H(s)$.

We recognize that closed-loop poles are located at the zeros of the system characteristic equation, or where $1+KG(s)H(s) = 0$. This condition is satisfied only when $KG(s)H(s) = -1$. Thus, if some point s_1 is on a branch of the root locus diagram, $KG(s_1)H(s_1) = -1$. Satisfaction of this condition requires that the following two conditions hold simultaneously, i.e.,

$$\angle G(s_1)H(s_1) = (2n+1)\,180^\circ \qquad (n = \text{integer}), \qquad K > 0$$

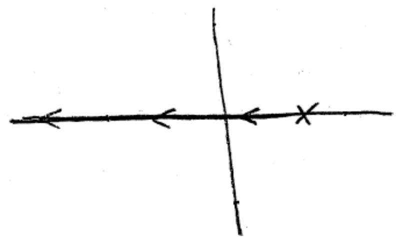
or

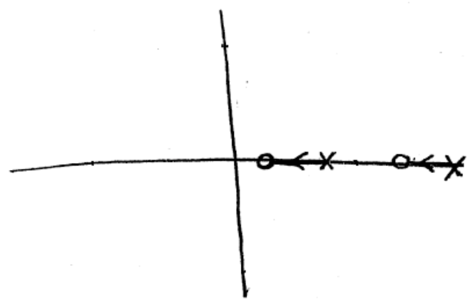
$$= (2n)\,180^\circ \qquad (n = \text{integer}), \qquad K < 0$$

and

$$|KG(s_1)H(s_1)| = 1$$

- The number of branches equals the number of poles of $G(s)H(s)$.
- Branches start at poles of $G(s)H(s)$ for $K = 0$ and end at zeros of $G(s)H(s)$ (either in the finite plane or at infinity) for large $|K|$.
- Branches on the real axis lie to the left of an odd number of poles and zeros of $G(s)H(s)$ for $K > 0$, and to the left of an even number of poles and zeros of $G(s)H(s)$ for $K < 0$.

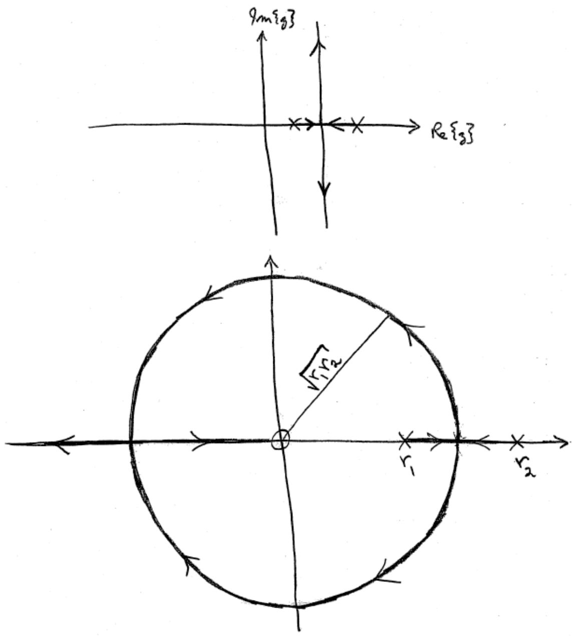




- Branches of the diagram must exist between any two real axis poles or zeros that satisfy the previously stated rule. Branches enter or leave the real axis at points where $\frac{d}{ds}G(s)H(s) = 0$. This point can be found by numerically maximizing or minimizing

$$M(\sigma) = \frac{\Pi(\sigma - p_i)}{\Pi(\sigma - z_i)}$$

where σ is the real location of either the breakaway point or the entry point, p_i is the location of the i^{th} open-loop pole, and z_i is the location of the i^{th} open-loop zero. It is almost always better to find the numerical extremum by direct evaluation rather than by differentiating and solving the resulting equation.



- For large values of $+K$, $P-Z$ branches go to infinity (P equals number of poles of $G(s)H(s)$, Z equals number of zeros of $G(s)H(s)$). These branches approach asymptotes that make angles of

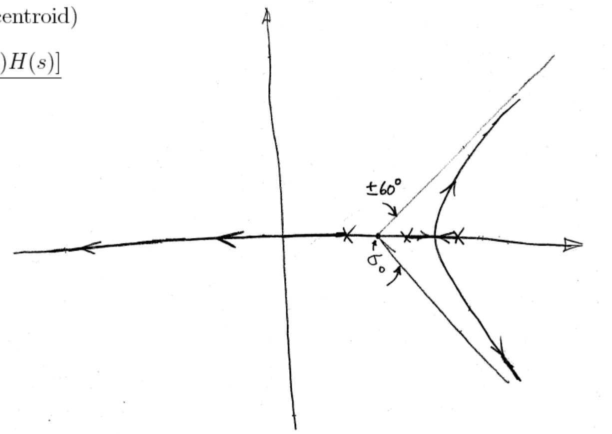
$$\frac{(2n+1)180^\circ}{P-Z} \qquad n = 0, 1, 2, \cdots, (P-Z-1)$$

with the real axis. For large negative values of K the angles of the asymptotes are

$$\frac{(2n)180^\circ}{P-Z} \qquad n = 0, 1, 2, \cdots, (P-Z-1)$$

Asymptotes intersect the real axis at a point σ_o (known as the centroid)

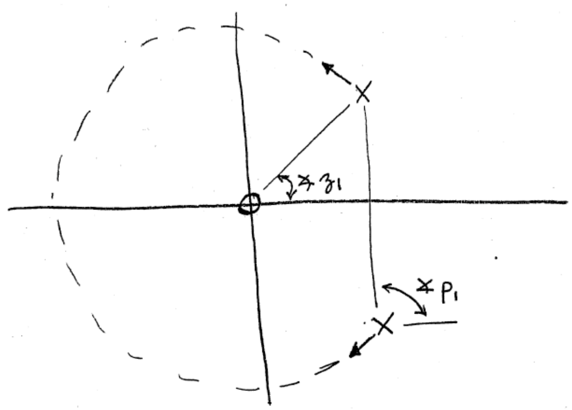
$$\sigma_o = \frac{\sum \text{Re[poles of } G(s)H(s)] - \sum \text{Re[zeros of } G(s)H(s)]}{P-Z}$$



$P-Z$	1	2	3	4	...
θ_n	π	$\pi/2$, $-\pi/2$	$\pi/3$, π, $5\pi/3$	$\pi/4$, $-\pi/4$, $3\pi/4$, $-3\pi/4$	...

- Near a complex pole or zero of $G(s)H(s)$ the angle of departure or entry is

$$\theta_p = [180^\circ \text{ for } K > 0;\ 0^\circ \text{ for } K < 0] + \sum \angle z - \sum \angle p$$
$$\theta_z = [180^\circ \text{ for } K > 0;\ 0^\circ \text{ for } K < 0] - \sum \angle z + \sum \angle p$$

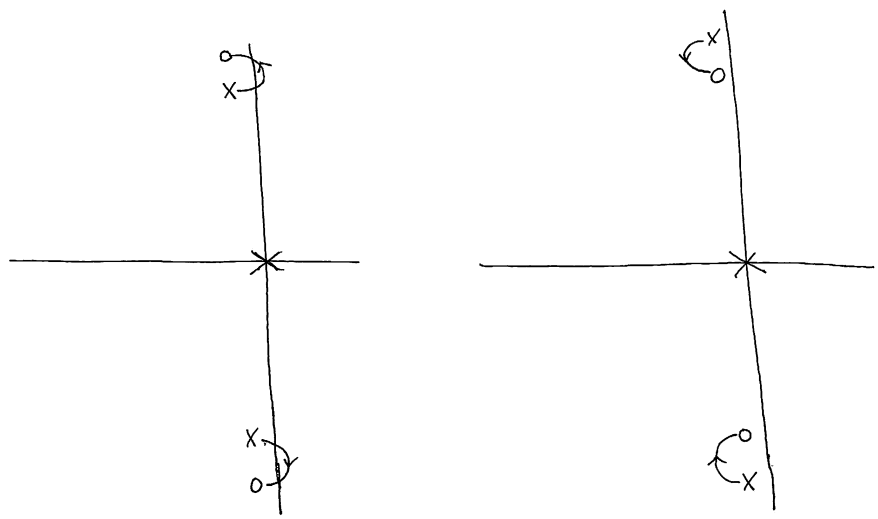


$\Rightarrow \theta_{p2} = 180^\circ + 45^\circ - 90^\circ = 135^\circ$

Similarly

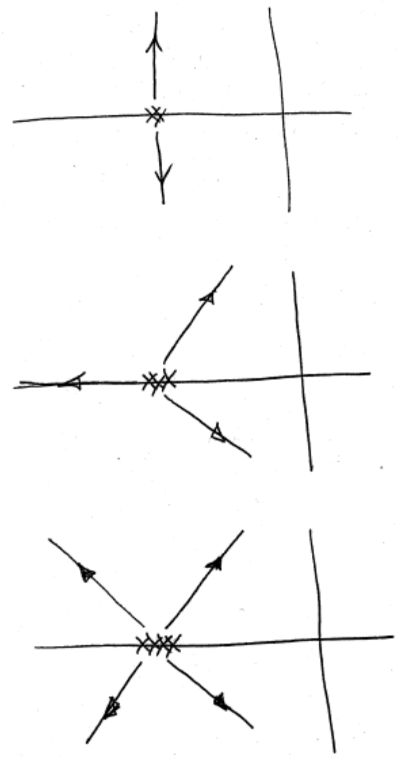
$\theta_{p1} = 180^\circ - 45^\circ + 90^\circ = 225^\circ$

Angle condition impacts notch filter design

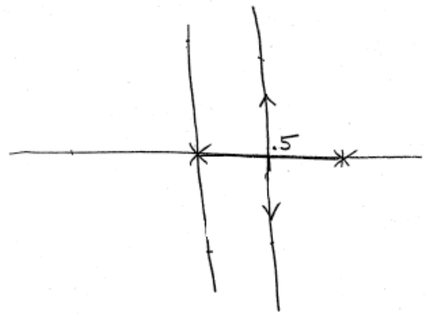


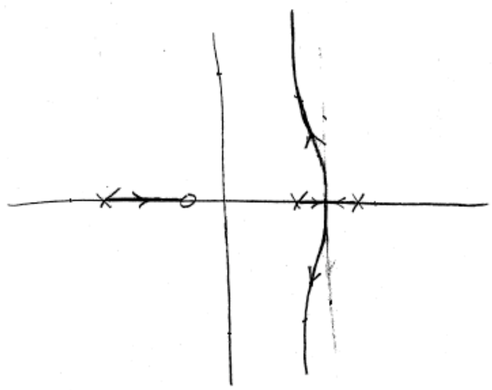
- The angles of departure (or entry) from (or to) multiple (order m) poles (or zeros) on the real axis are

$$\frac{(2n+1)\,180^\circ}{m} \quad \text{or} \quad \frac{2n\,180^\circ}{m}, \qquad n = 0, 1, 2, \cdots, m$$

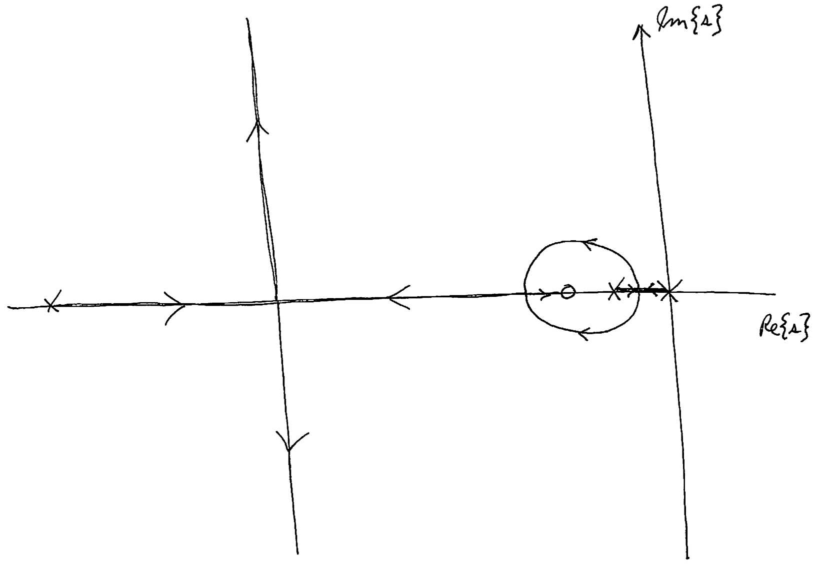


- If the number of poles of $G(s)H(s)$ is greater than or equal to two plus the number of zeros of $G(s)H(s)$, the average distance of the closed-loop poles from the imaginary axis remains constant (Grant's rule).



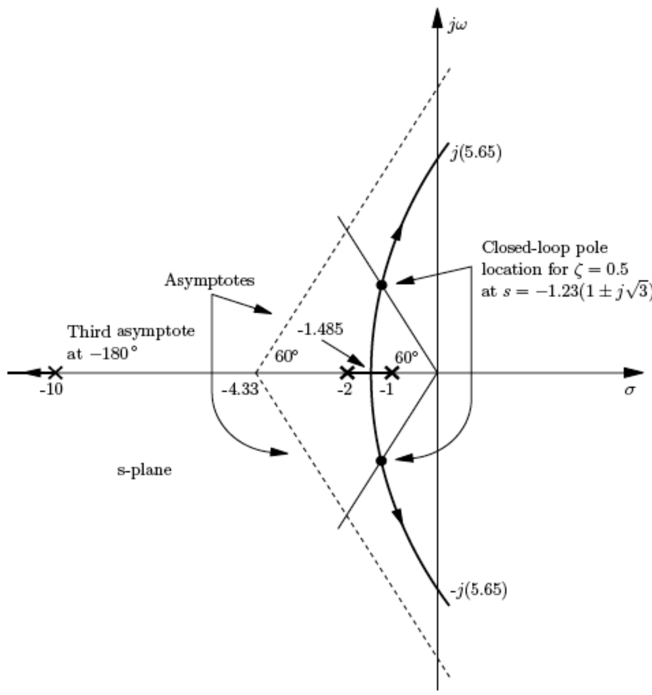


- Ignore remote poles and zeros of $G(s)H(s)$ when plotting loci near the origin. Also combine poles and zeros of $G(s)H(s)$ near origin when finding loci at large $|s|$.

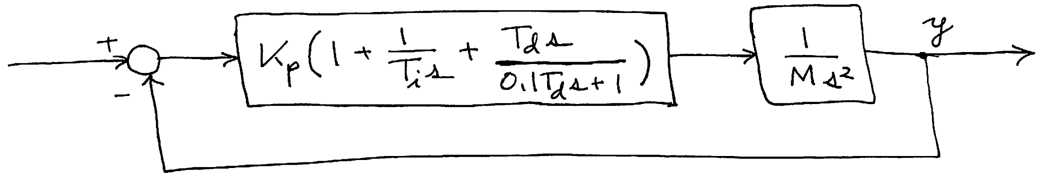


As a simple example that illustrates the use of some of these rules, consider

$$KG(s)H(s) = \frac{K}{(s+1)(0.5s+1)(0.1s+1)}$$

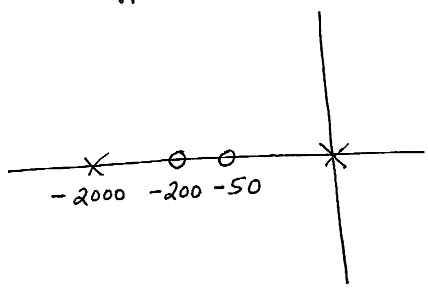


PID Example from earlier, with 10 kg mass

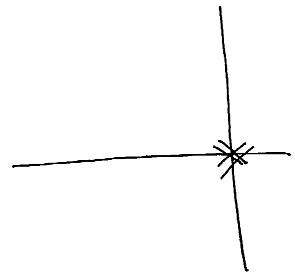


$K_p = 2.5 \times 10^6$; $T_i = 20$ msec ; $T_d = 4$ msec ; $M = 10$ kg

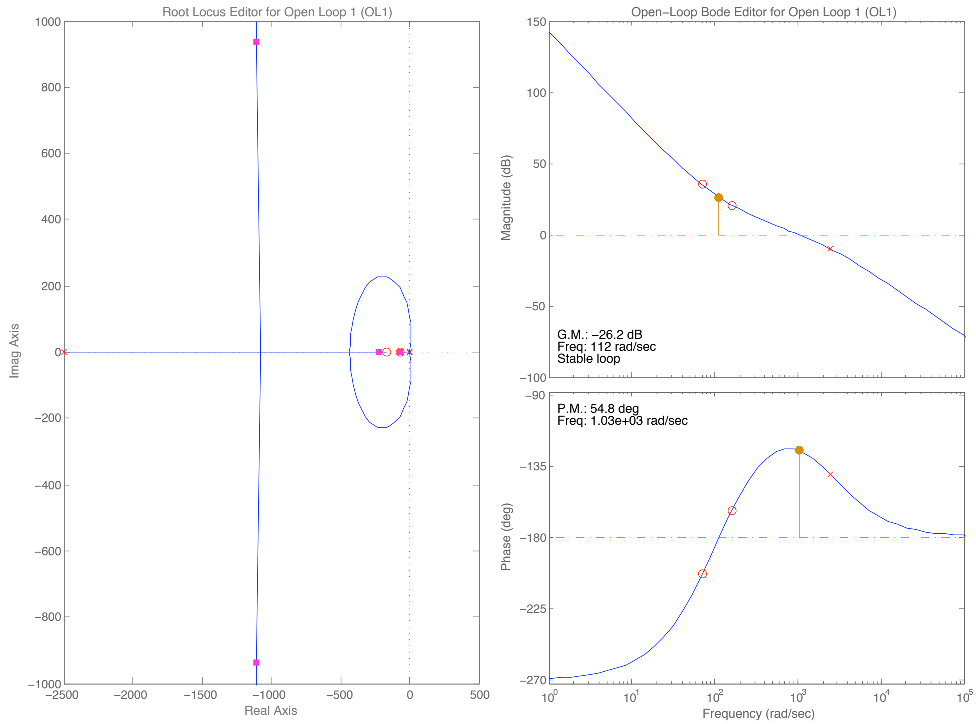
Controller poles/zeros (approx.)



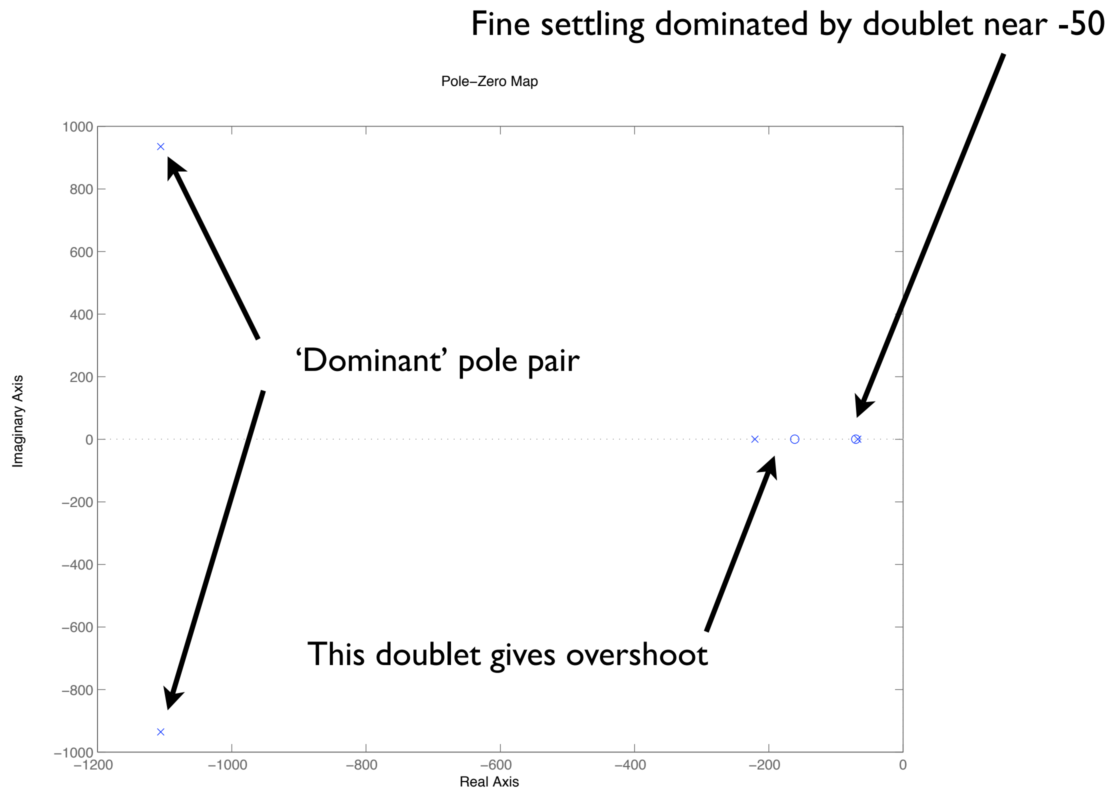
Plant poles

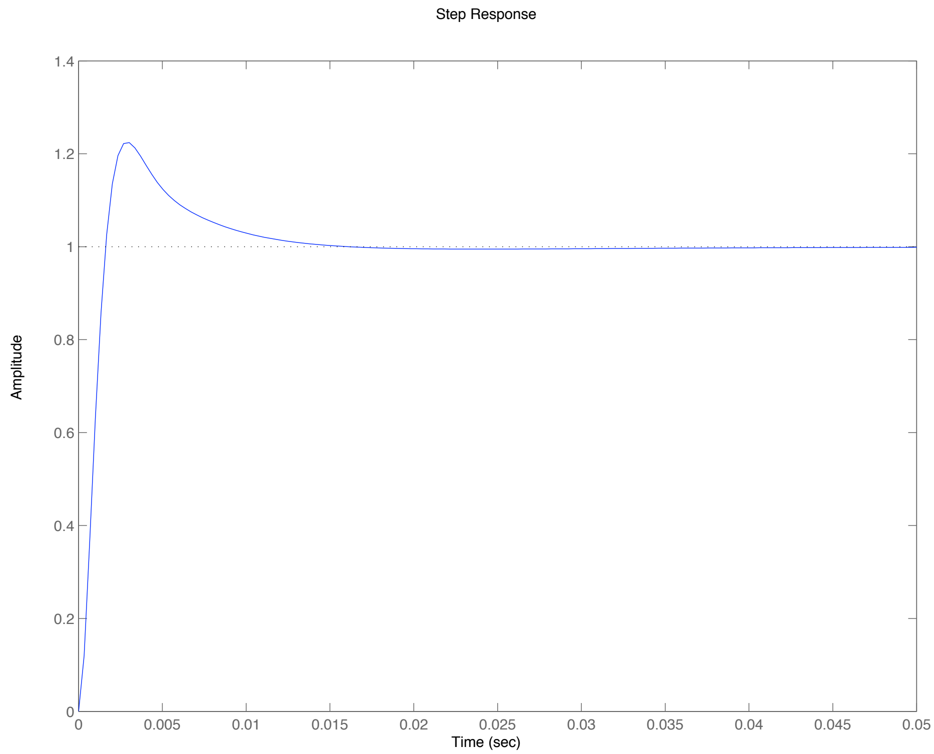


Use sisotool.m to get root locus and loop shape plots; see Matlab routine pidloop.m

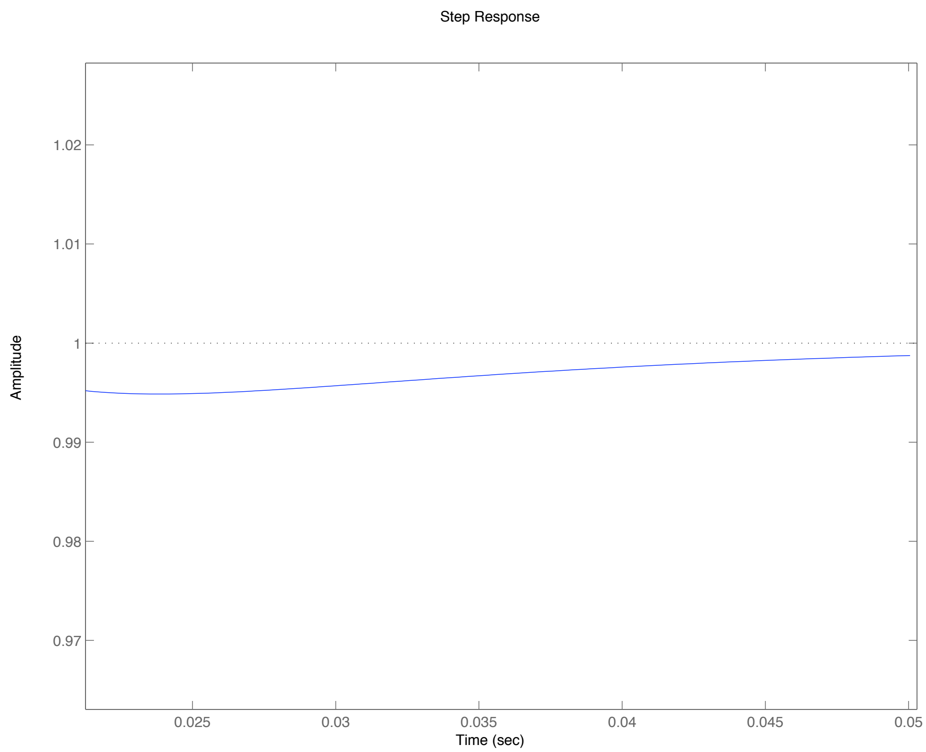


Closed-loop poles and zeros




Step Response
Amplitude
Time (sec)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0.04
0.045
0.05

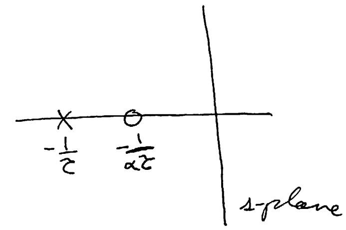
20 msec time constant in fine settling due to pole at about -50, which was attracted in root locus sense to the zero in that vicinity



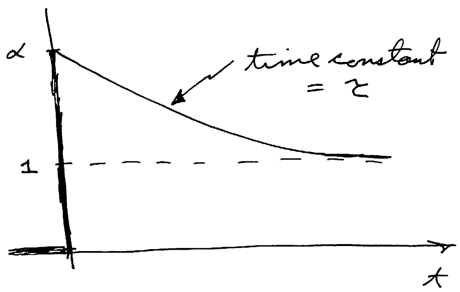
Doublets:

Lead: $H(s) = \dfrac{\alpha \tau s + 1}{\tau s + 1}$

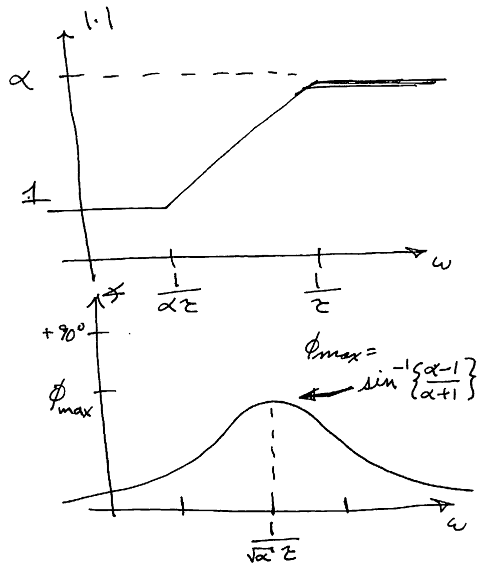
$\alpha > 1$



Step response:



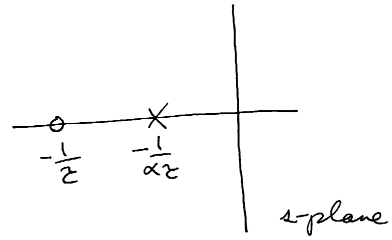
Bode plot



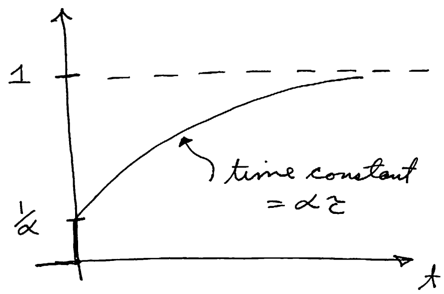
Lag

$$H(s) = \frac{\tau s + 1}{\alpha \tau s + 1}$$

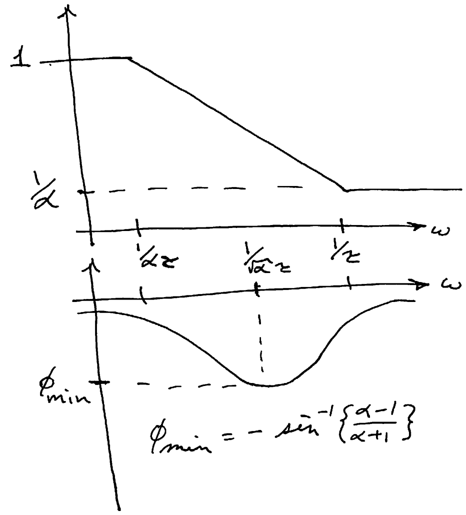
$\alpha > 1$



Step response:

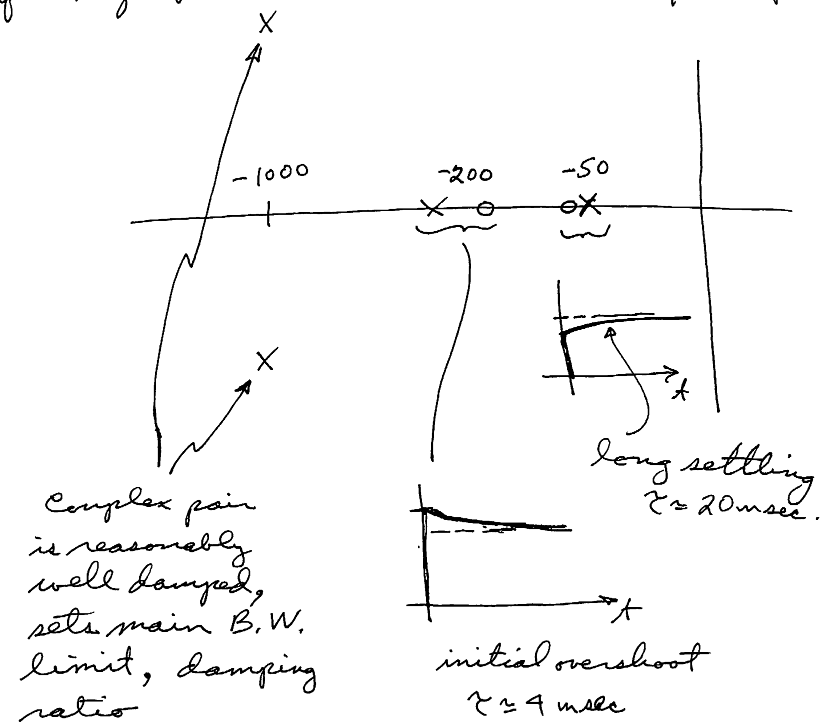


Bode plot:



$$\phi_{min} = -\sin^{-1}\left\{\frac{\alpha - 1}{\alpha + 1}\right\}$$

Doublets in C.L. poles/zeros of PID loop account for significant characteristics of response



Now, root locus shows the location of closed-loop <u>poles</u> as a function of loop gain K.

- Does <u>not</u> give closed-loop zeros.
- zeros of C.L. <u>do</u> <u>not</u> <u>move</u> with gain K.

(?) Where are the closed-loop zeros?

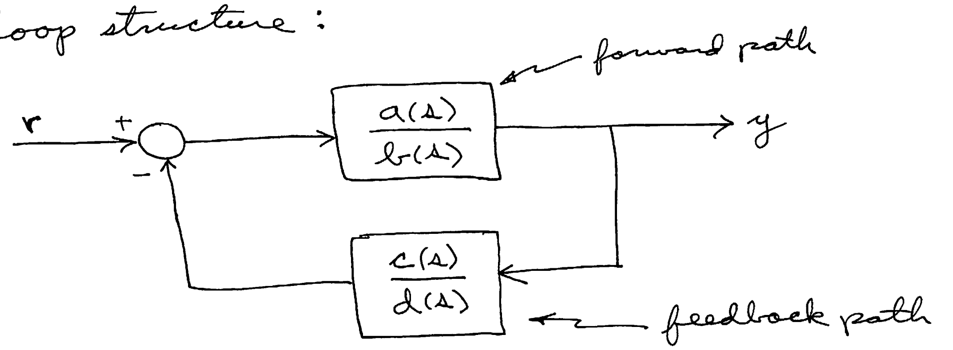
Answer: 1) zeros of the forward path

2) Poles of the feedback path

- In our PID example, all loop zeros are in F.P. ⇒ are also C.L. zeros
  - no dynamics in feedback path, in particular no poles in feedback path ⇒ no zeros from this source.

<u>SO</u>: The zeros shown on the R.L. plot are also the C.L. zeros of the PID controlled loop.

Loop structure:



Closed loop is

$$\frac{Y(s)}{R(s)} = \frac{a/b}{1 + \frac{ac}{bd}} = \frac{ad}{bd + ac}$$

C.L. zeros: $a(s)d(s) = 0 \Rightarrow$ { zeros of forward path, poles of feedback path.

C.L. poles: $b(s)d(s) + a(s)c(s) = 0$

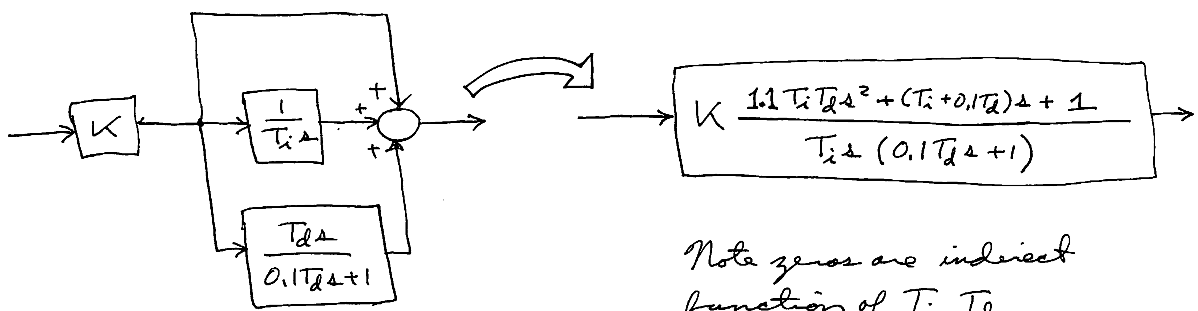
depends upon <u>all</u> singularities in loop

Classical 3-term PID controller is <u>not</u> a good way to implement, in general

$$G_c(s) = K\left(1 + \frac{1}{T_i s} + \frac{T_d s}{0.1 T_d s + 1}\right)$$

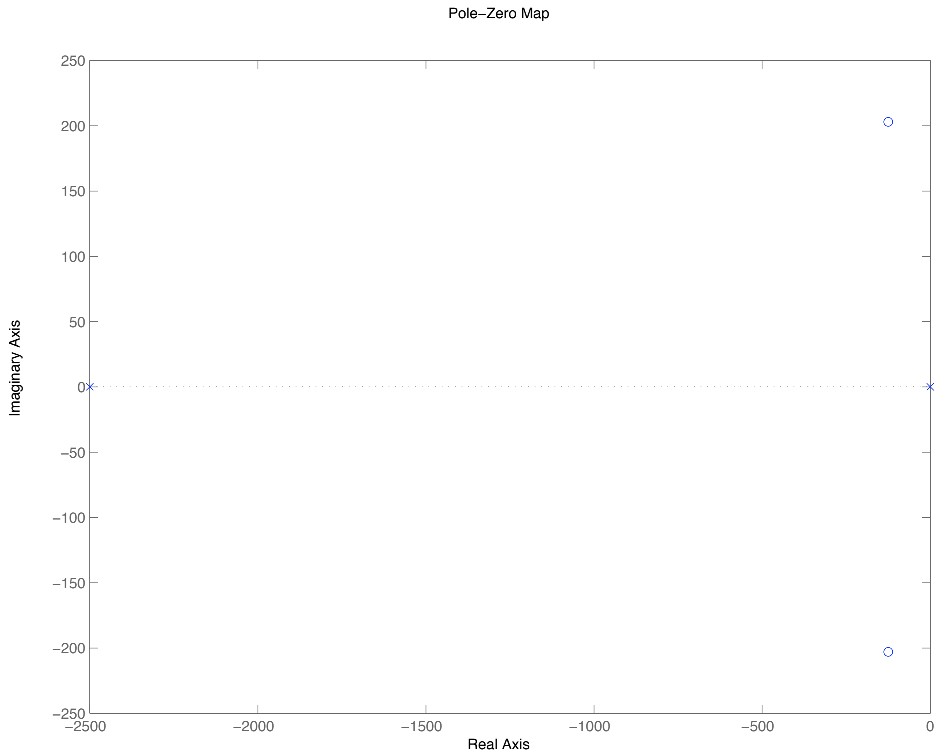
Why not?

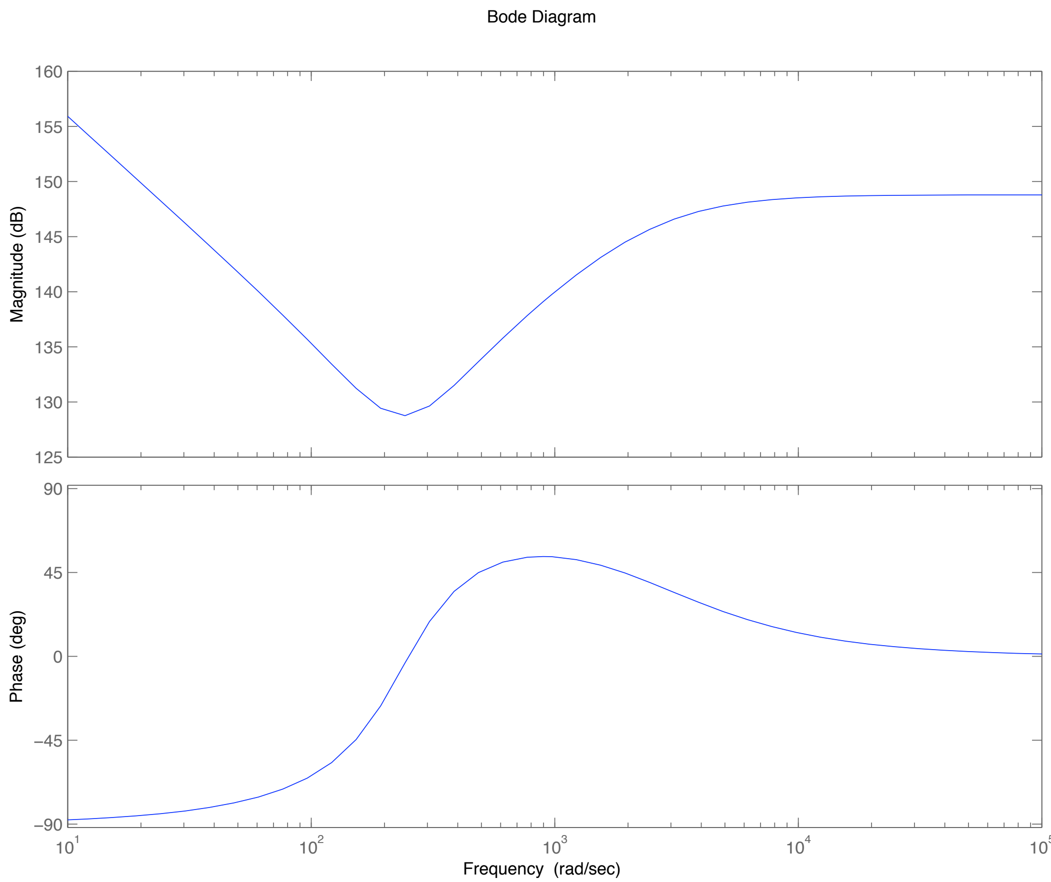
In this parallel realization, zeros are not directly specified



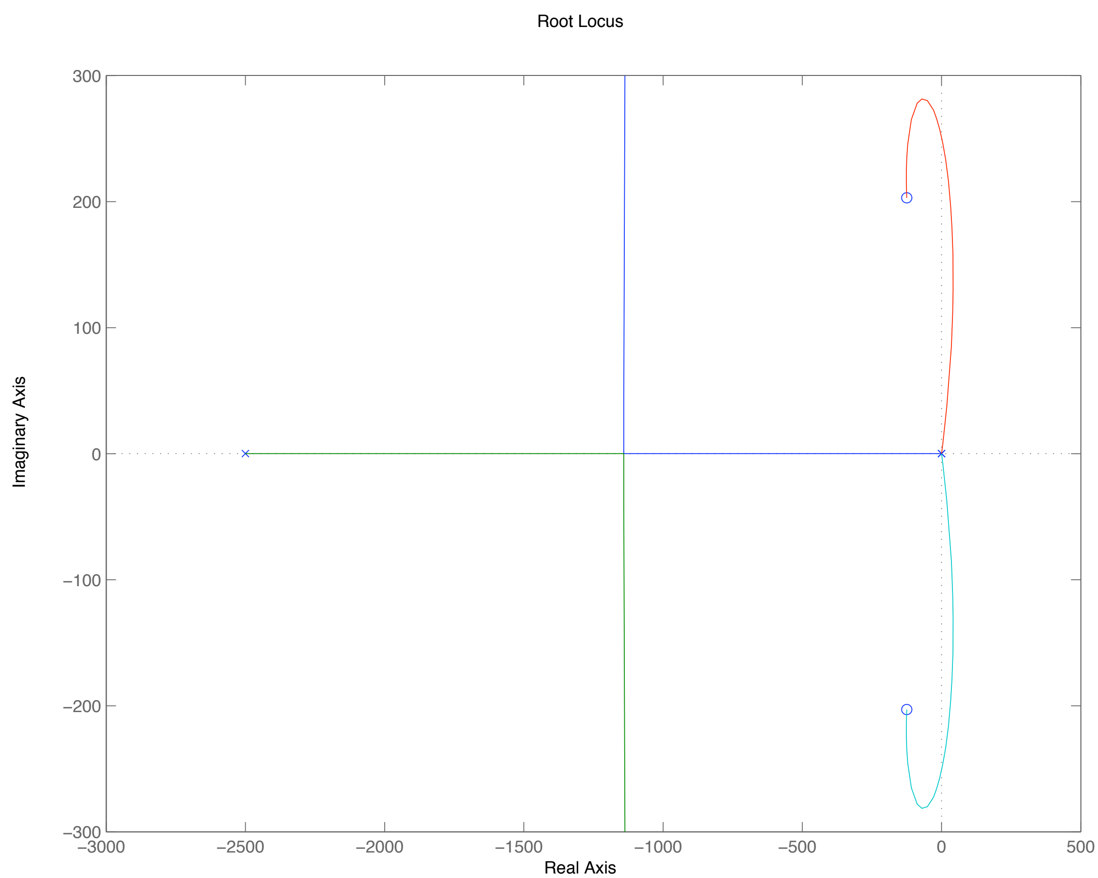
Note zeros are indirect function of T_i, T_d (may even end up complex!)

For Ti = Td = 4 msec:



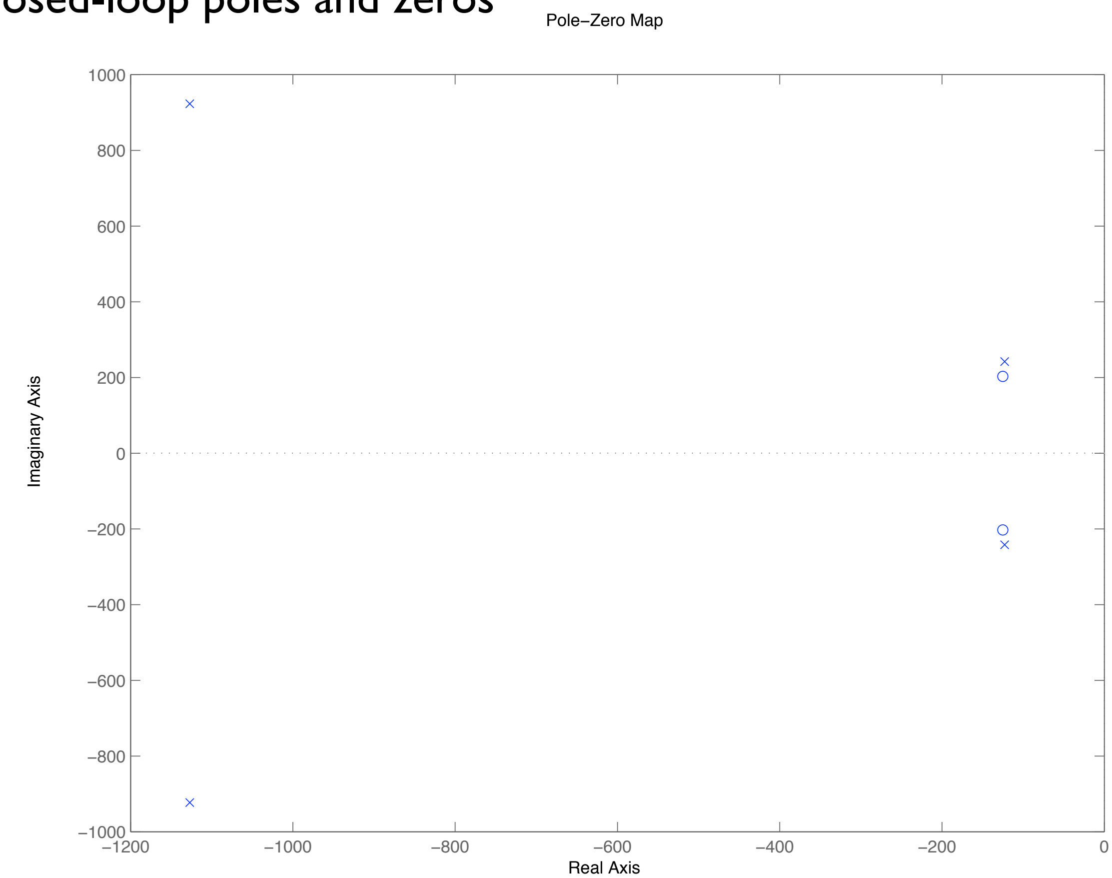


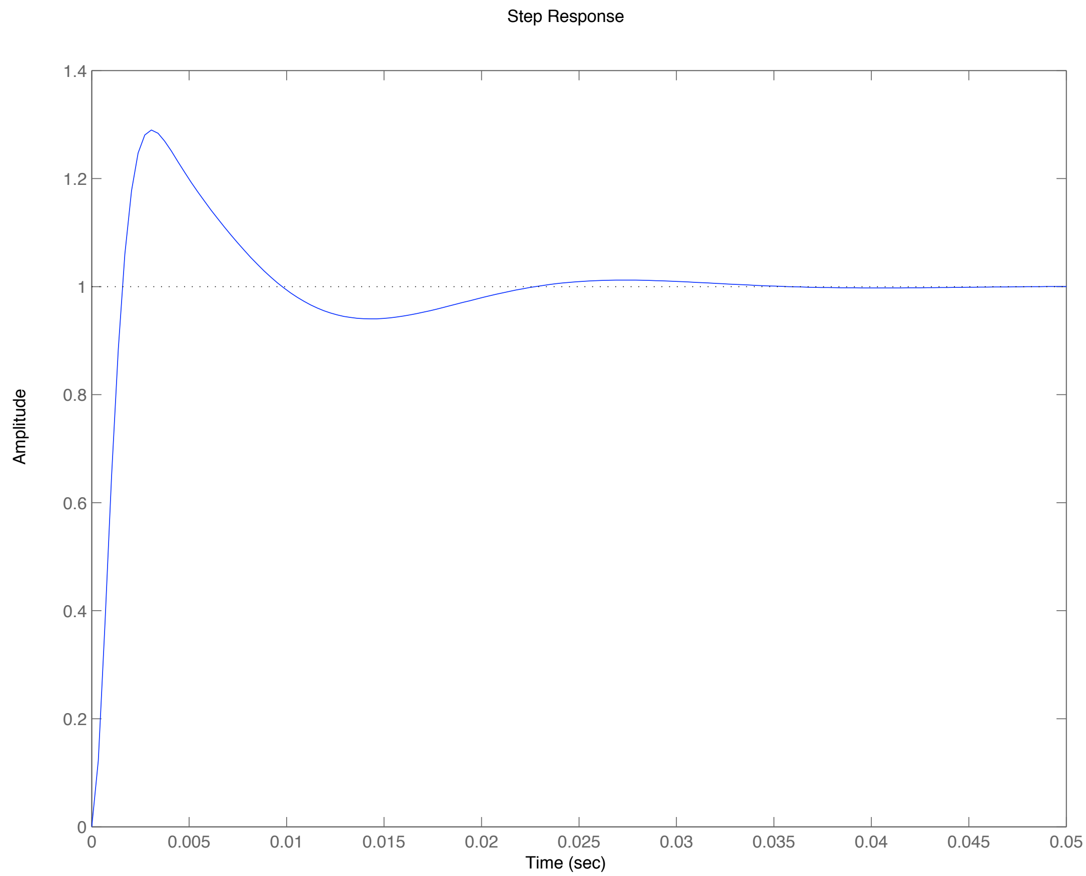
Bode Diagram
Magnitude (dB)
160
155
150
145
140
135
130
125
Phase (deg)
90
45
0
−45
−90
10^1
10^2
10^3
10^4
10^5
Frequency (rad/sec)

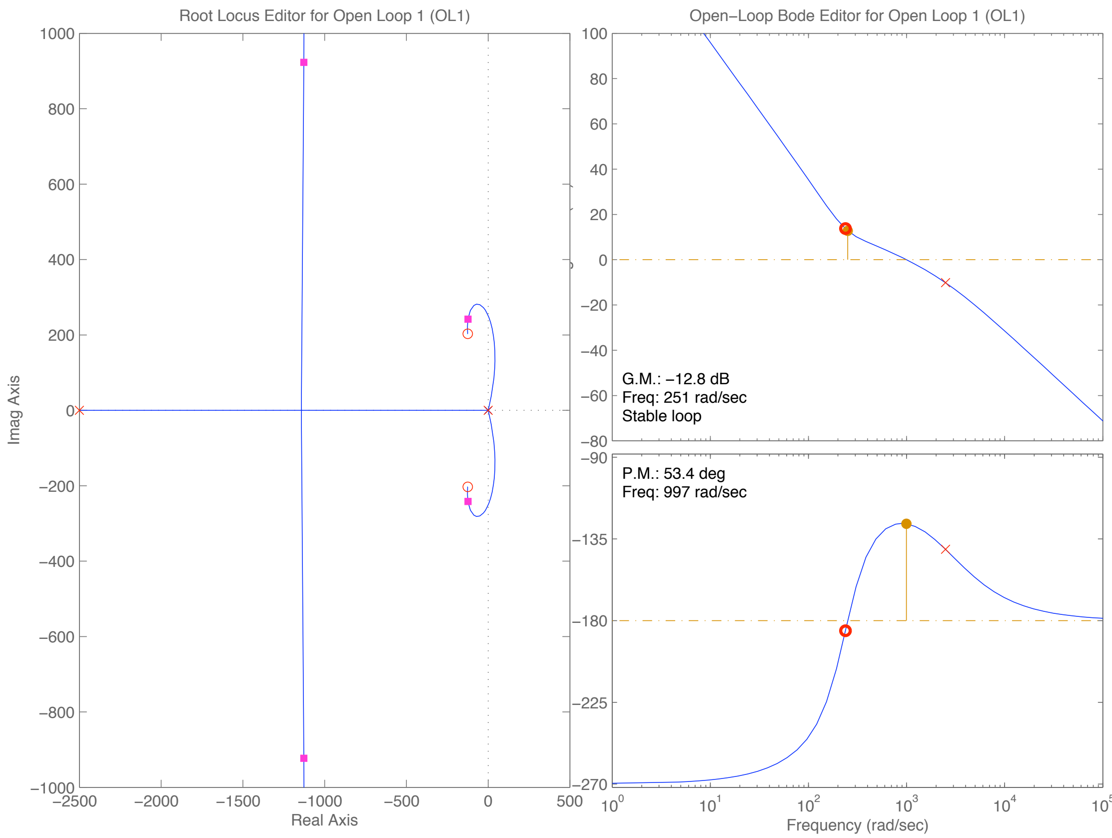


Root Locus
Imaginary Axis
Real Axis
300
200
100
0
−100
−200
−300
−3000
−2500
−2000
−1500
−1000
−500
0
500

Closed-loop poles and zeros



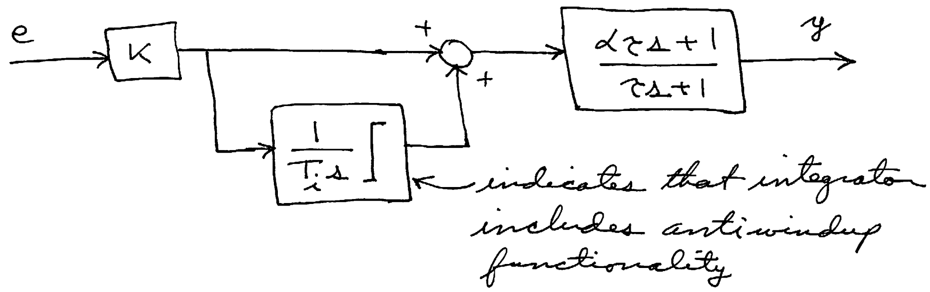

Step Response
Amplitude
Time (sec)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0.04
0.045
0.05



Root Locus Editor for Open Loop 1 (OL1)
Imag Axis
Real Axis
Open–Loop Bode Editor for Open Loop 1 (OL1)
G.M.: −12.8 dB
Freq: 251 rad/sec
Stable loop
P.M.: 53.4 deg
Freq: 997 rad/sec
Frequency (rad/sec)

In some (rare) circumstances, complex-valued compensator zeros might be useful, but better not to end up there accidentally.

A better PID structure is the series/parallel realization:



$$G_c(s) = K\left(\frac{T_i s + 1}{T_i s}\right)\left(\frac{\alpha \tau s + 1}{\tau s + 1}\right)$$

note that zero and pole locations are explicitly determined. Also, anti-windup easily added.